EQUATIONS OF THE THIRD DEGREE.

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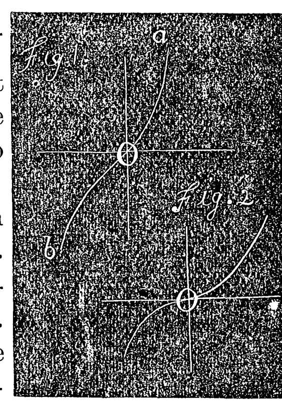
In the equation $X^3 \pm AX^2 \pm BX \pm C = Y$, A, B and C will be considered as representing real and not imaginary quantities. Let X be any abscissa and Y the ordinate corresponding to it. We shall then have a curve of which $X^3 \pm AX^2 \pm BX \pm C = Y$ is the equation. The origin can be so taken as to reduce the equation to the form $x^3 \pm px \pm q = y$. This will not alter the form of the curve at all. Again the constant q, the absolute term, can be omitted without changing the form of the curve. Hence we need consider only such equations as $x^3 \pm px = y$.

CASE I. $x^3 + px = y$. The curve will be of the general type seen in fig. 1.

1. It will pass through the origin.
2. It will have two infinite branches.
3. For every $+$ value of x, there will be a corresponding $+$ value of y; and for every equal $-$ value of x there will be an equal $-$ value of y. Hence every chord passing through the origin is bisected there, and the origin may be called the centre of the eurve.
4. Since $\dfrac{dy}{dx} = 3x^2 + p$, when $x = 0$, the trigonometrical tangent of the angle the curve tangent makes with the axis of abscissas $= p$. As the sign of this is $+$, the angle is less than $90°$, so long as p is finite.



The tangent to the curve at other points than the origin will approach a vertical direction, i. e. a direction parallel to the axis of ordinates, whether x be $+$ or $-$, because in either case x^2 will be $+$. Since $3x^2 + p$ increases with every increase of x, the curve is concave to the axis of ordinates both below and above the origin.

5. It is evident, even without a second differentiation, that the minimum value of $3x^2 + p$ is p; $\therefore$ there is a point of inflexion at the origin.

Points of inflexion mark maximum and minimum values of angles where the trigonometrical tangent does not change its sign. The ordinary method, however, is very simple; $\dfrac{d^2y}{dx^2} = 6x = 0$, $\therefore x = 0$. As $6x$ is of the 1st degree there can be only one point of inflexion, and there must be one even though p were imaginary since p disappears at the 2nd differentiation.

6. As $p > 0$, the minimum angle of inclination is > 0; $\therefore$ the curve never becomes parallel to the axis of abscissas.

7. An indefinite number of curves may be drawn of this class.

CASE II. Let $p = 0$, then $x^3 = y$. (See fig. 2.)

1. As p has only one value here, only one curve can be described.

2. $\frac{dy}{dx} = 3x^2 = 0$ when $x = 0$; $\therefore$ the curve is parallel to the axis of abscissas at the origin. The point of inflexion is there, also. In most other particulars the curve resembles those of the 1st class.

CASE III. $x^3 - px = y$. (See fig. 3.)

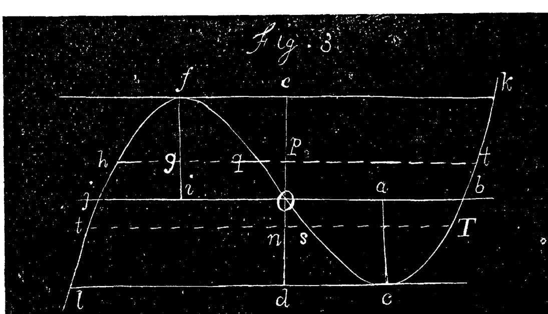
1. $\frac{dy}{dx} = 3x^2 - p$. Therefore at the origin the curve makes with the axis of abscissas an angle whose tangent $= -p$. As x increases on the + side or on the —, it will reach a point where $3x^2 = p$, or $3x^2 - p = 0$; $\therefore$ there are two points at which the curve is parallel to the axis of abscissas, one for $+x = \sqrt{(\frac{1}{3}p)}$, and the other for $-x = \sqrt{(\frac{1}{3}p)}$. For greater values of x, either positive or negative, $3x^2 - p$ is positive, and, as before, the curve is concave towards the axis of ordinates.

2. The point of inflexion is at the origin, making a maximum value of the tangent.

3. An indefinite number of lines of this class can be drawn.

TRANSFORMATION OF CO-ORDINATES.

1. The origin can be placed higher up the page by annexing $-q$ to the left hand member of an eq'ion, or lower by annexing $+q$. Thus if the eq'n be $x^3 \pm px = y$ the origin is at the point of inflexion. It may be moved 6 spaces higher by making the eq'n $x^3 \pm px - 6 = y$, or 6 spaces lower by $x^3 \pm px + 6 = y$. The origin remains on the axis of ordinates.



2. The origin can be removed to the right by substituting $x' + a$ for x; to the left, by substituting $x' - a$ for x. Or the same result can be obtained by Horner's method of detached coefficients, dividing by $+a$, or $-a$.

ROOTS OF THE THREE FORMS.

1. The form $x^3 + px = y$ has only one real root, and can have no more real roots when the origin is removed by any of the transformations. This

root is found by making $y = 0$. The curve at that point touches or cuts the axis of abscissas. The root may be found by Cardan's Rule.

2. The 2nd form, $x^3 = y = 0$, has three equal roots so long as the origin remains on the axis of abscissas.

$x^3 = 0$ has 3 roots, each $= 0$, $(x-a)^3 = 0$ has 3 roots, each $= +a$, and $(x + a)^3 = 0$ has 3 roots, each $= -a$. But $x^3 \pm q = 0$ has one real root and two imaginary ones; however the origin may be changed to right or left, the number of the real and the imaginary roots will be the same, one real and two imaginary.

In the former case $q = \mp a^3$; $\therefore$ the distance to the right or left of the origin at which the curve cuts the axis of abscissas is equal the cube root of the distance above or below the origin at which it cuts the axis of ordinates.

In $x^3 \pm q = 0$, $x = \sqrt[3]{\mp q}$ gives the one real root. The imaginary roots can be obtained by the ordinary methods.

3. The 3rd form, $x^3 - px = 0$, and its transformations, will always have at least one real root. It will have 3 real roots, 2 of them being equal, when the origin is removed from the point of iflexion to *e* or *d*, or to any pos'n on the line *kef* or *cdl*, these lines being parallel to the axis of abscissas and tangent respestively to the max. and min. points of the curve.

We have already found $Oa = \sqrt{(\frac{1}{3}p)}$, $Oi = -\sqrt{(\frac{1}{3}p)}$. If $q > Oe$, or $-q > -Od$, numerically, there will be only one real root. If $q < Oe$, or $-q < Od$, numerically, there will be 3 real and unequal roots. If $q = 0$, one root will be 0, one will $= \sqrt{(\frac{1}{3}p)}$, and one will $= -\sqrt{(\frac{1}{3}p)}$.

Example. $x^3 - 12x = 0$. The roots are $0, +\sqrt{(12)}, -\sqrt{(12)}$. It is required to ascertain what absolute term shall be added or subtracted so that there shall be two equal roots. Here $p = 12$, $\therefore \sqrt{(\frac{1}{3}p)} = 2$. Substituting this in $x^3 - 12x$, we get $8 - 24 = -16$. $\therefore x^3 - 12x + 16 = 0$ has the roots $+2$, $+2$, and -4. $x^3 - 12x - 16 = 0$ has the roots $-2, -2, +4$. In general $x^3 - px + \sqrt{(\frac{4}{27}p^3)}$ or $x^3 - px + \frac{2}{3}p\sqrt{(\frac{1}{3}p)}$ will have the roots $+\sqrt{(\frac{1}{3}p)}$, $+\sqrt{(\frac{1}{3}p)}$, and $-2\sqrt{(\frac{1}{3}p)}$; and $x^3 - px - \sqrt{(\frac{4}{27}p^3)}$ will have the roots $-\sqrt{(\frac{1}{3}p)}$, $-\sqrt{(\frac{1}{3}p)}$, $+2\sqrt{(\frac{1}{3}p)}$. These roots are identical with those obtained by Cardan's Rule, which applies to all cases in which there is only one real root, and also to those in which there are two equal real roots. But it has no sway over the charmed area between the lines *kef* and *cdl*. Just here the "irreducible case" lies.

4. Our attention must now be directed to the curve *kcofl*. But as this curve is symmetrical, it will be sufficicnt in most respects to consider only one half of it, as *kco*.

There is no trouble in finding the roots when the origin is at *O*, or at *d*; as we have seen. But suppose it is placed at *n*, any point between *O* and *d*.

Then we are to determine the lengths of the lines nT, ns and nt. Let $nT = r$, $ns = s$, $nt = t$. Then, by general theory of equations, $r+s-t=0$, $rs-rt-st=-p$ and $rst=q$. When, then, the equation is $x^3-px+q=0$ and there are three unequal roots, two of them will be $+$, and the other $-$.

It is readily seen that if the origin be removed to p in the figure, the curve will cut the axis of ordinates below the new origin; $\therefore$ q is negative. Therefore in the equation $x^3-px-q=0$, if there are three unequal real roots two of them are $-$ and the other $+$.

If the axis of abscissas be removed from n upward, ns will gradually decrease to zero and nT increase to Ob; therefore the ratio $s \div r$ has for one of its limits zero. If the axis be removed downward, ns will increase and nT will diminish until they both become dc; therefere the other limit of the ratio $s \div r$ is unity. Let a denote this ratio. It may have any value between 0 and 1. $s=ar$, $r+s=t$; $\therefore$ $t=r+ar$, $ar^2-r^2-ar^2-ar^2-a^2r^2=-p$.

$$\therefore\ r^2=\frac{p}{1+a+a^2},\quad ar^3+a^2r^3=q,\quad \therefore\ r^3=\frac{q}{a+a^2}.$$

$$r^6=\frac{p^3}{(1+a+a^2)^3}=\frac{q^2}{(a+a^2)^2};\quad \therefore\ \frac{p^3}{q^2}=\frac{(1+a+a^2)^3}{(a+a^2)^2}.$$

We may now form a table taking the different values of a from .001 by intervals of one thosandth up to 1. As p and q are known, $p^3 \div q^2$ is known, and $\therefore$ $(1+a+a^2)^3 \div (a+a^2)^2$ is known, and we can find the value of a just as from a given natural sine we find the corresponding arc. Then

$$r=\sqrt{\left(\frac{p}{1+a+a^2}\right)},$$

$s=ar$, and $t=r+ar$. Or $r=\sqrt[3]{\left(\dfrac{q}{a+a^2}\right)}$.

A better mode of procedure is to take the logarithms of both members. Thus $\qquad 3\log p-2\log q=3\log(1+a+a^2)-2\log(a+a^2)$.

Before illustrating by an example, it may be well to get the maximum and minimum values of $(1+a+a^2)^3 \div (a+a^2)^2$ or of

$$3\log(1+a+a^2)-2\log(a+a^2).$$

We find $2a^3+3a^2-3a-2=0$, or $a^3+\frac{3}{2}a^2-\frac{3}{2}a-1=0$. Evidently one of the roots is 1. The others are $-\frac{1}{2}$ and -2. Taking both the abscissas on the right of the axis ed, we interpret $a=1$ by $dc=dc$; $a=-\frac{1}{2}$ brings in the negative $ef=\frac{1}{2}ek$, and $a=-2$, $dl=2dc$. Confining ourselves to the case in which $a=1$, we reach the lowest point of the curve at c. Then $\dfrac{(1+a+a^2)^3}{(a+a^2)^2}=\dfrac{27}{4}$, which is the smallest admissible value also of $\dfrac{p^3}{q^2}$ when there are 3 real roots. This corresponds precisely with the results of Cardan's Rule.

If $a > 1$, which is contrary to the supposition, we merely exchange s for t and r for s. The substitution of $-\frac{1}{2}$ or -2 for a will give the same value of the fraction $\frac{27}{4}$.

It is plain then that our present method is applicable within the parallels *kef* and *cdl*, as Cardan's is without them, while they both give the same roots for the points f and c on those parallels.

Let us now take a common example, $x^3-7x+7=0$. Here $-p=-7$, $+q=7$, $\log(p^3 \div q^2)=\log 7 = .8450980$. Referring to the table we see that

when $a = .801$, $\log[(1+a+a^2)^3 \div (a+a^2)^2] = .84526533$.

When $a = .802$, $\log \qquad = .84508693$.

therefore .802 is a pretty close approximation. But we can come nearer by a proportion:

$$\begin{array}{r r}
.84526533 & .84526533 \\
\underline{.84508693} & \underline{.84509800} \\
17840 : & 16733 :: .001 : .000937948. \\
\therefore & a = .801937948.
\end{array}$$

To find $r=\sqrt{\left(\frac{p}{1+a+a^2}\right)}$; because $\log r = \frac{1}{2}(\log p - \log(1+a+a^2)$, it is necessary to square a; but as a is too large when obtained by the process in full, we multiply as follows:

$$\begin{array}{r}
.801937948 \\
\underline{.801937948} \\
640 \\
3204 \\
72171 \\
561351 \\
2405811 \\
72174411 \\
80193794 \\
\underline{.64155035840} \\
.64310447222
\end{array}$$

$\therefore\ 1+a+a^2 = 2.44504242.$

$$\begin{array}{l l}
\log p & = .8450980 \\
\log(1+a+a^2) & = \underline{.3882862} \\
\therefore 2\log r & = .4568118 \\
\log r & = .2284059; \ \therefore r = 1.692021484.
\end{array}$$

By Sturm and Horner $\qquad r = 1.692021471,$

error $\ .000000013.$

This is more exact however than we can ordinarily be.

As a convenient test we may take $r^3 = q \div (a+a^2)$, $\therefore 3\log r = \log q - \log(a+a^2)$; $\therefore r = 1.6920212$, not so close.

One marked advantage of the method is the ease of finding the 2nd root, s.

$$
\begin{array}{llll}
\text{Log}\, r = 0.2284059 & & & \\
\text{"} \quad a = \underline{1.9041407} & & & \\
\quad .1325466. & \therefore s = & 1.356896250; & \\
\text{by Sturm and Horner} & & \underline{1.356895867} & \\
 & \text{error} & .000000383. &
\end{array}
$$

The 3rd root is negative and is equal to the numerical sum of the other two roots. The error is about .0000004. It should be observed that if a, as actually found, is too large, $r = p \div (1 + a + a^2)$ is too small; but then ar tends to right the second root, and not to increase the error; and vice versa.

For another example take $x^3 - 3x - 1 = 0$. $\text{Log}\, p^3 - \log q^2 = \log 27 = 1.4313638$, log q being 0. Two roots are negative.

$$a = .226 \text{ gives } \quad 1.43345238;$$
$$a = .227 \text{ gives } \quad 1.43039080.$$

By proportion we get $a = .22668219$;

$$1 + a + a^2 = 1.27806700.$$

$$
\begin{array}{lll}
\text{Log}\, p = .4771213, & & \\
\log(1 + a + a^2) = \underline{.1065536,} & & -1.532088886 \\
\quad .3705677 \div 2 = .18528385; & \therefore t = & -1.532088516 \\
 & & \text{error } .000000370.
\end{array}
$$

As before, I compare with the value found by Sturm and Horner, which I have not verified but suppose to be correct. The log. of 1.532088886 is

$$
\begin{array}{ll}
 & .18528385, \\
 & \underline{.18528385,} \\
\text{error} & .00000010.
\end{array}
$$

$$
\begin{array}{lll}
\text{Value of } s = at \text{ is} & & -.347297000; \\
\text{by Sturm and Horner} & & \underline{-.347296355,} \\
 & \text{error} & .000000645. \\
r = s + t & = & +1.879385516; \\
\text{by Sturm and Horner} & & \underline{+1.879385242,} \\
 & \text{error} & .000000274.
\end{array}
$$

By comparing in this way with the results of Horner more particularly, we can learn how to improve this method practically. Since a as rigorously found is always too large, and I have been using Vega's 7-place logarithms, it will be observed that, in subtracting in the first example, .8450980 from .84526533 the last 3 is retained in the remainder 16733. The effect is to diminish a slightly. The abbreviation in squaring .801937948 makes a a little too small. Squaring in full gives

$$\log r = .228405763262.$$

$$
\begin{array}{ll}
\text{Employed above} & .2284059; \\
\text{by Sturm and Horner} & .228405896576.
\end{array}
$$

TABLE.

a	$3 \log \alpha - 2 \log \beta$	a	$3 \log \alpha - 2 \log \beta$	a	$3 \log \alpha - 2 \log \beta$
.01	4.0044504	.34	1.1719586	.67	0.8807489
02	3 4070508	35	1 1553623	68	0 8770654
03	3 0597325	36	1 1395263	69	0 8735639
04	2 8149164	37	1 1244086	70	0 8702383
05	2 6263477	38	1 1099713	71	0 8670826
06	2 4734207	39	1 0961787	72	0 8640913
07	2 3451407	40	1 0829978	73	0 8612582
08	2 2349420	41	1 0703975	74	0 8585783
09	2 1385877	42	1 0583499	75	0 8560465
10	2 0531836	43	1 0468278	76	0 8536580
11	1 9766634	44	1 0358067	77	0 8514079
12	1 9075000	45	1 0252635	78	0 8492921
13	1 8445332	46	1 0151758	79	0 8473057
14	1 7868588	47	1 0055233	80	0 8454444
16	1 7337605	48	0 9962871	81	0 8437049
16	1 6846586	49	0 9874490	82	0 8420827
17	1 6390794	50	0 9789914	83	0 8405743
18	1 5966287	51	0 9708992	84	0 8391764
19	1 5569765	52	0 9631565	85	0 8378848
20	1 5198427	53	0 9557489	86	0 8366967
21	1 4849872	54	0 9486628	87	0 8356087
22	1 4522036	55	0 9418858	88	0 8346182
23	1 4213126	56	0 9354055	89	0 8337214
24	1 3921466	57	0 9292099	90	0 8329157
25	1 3645979	58	0 9232884	91	0 8321983
26	1 3385138	59	0 9176304	92	0 8315668
27	1 3137961	60	0 9122259	93	0 8310183
28	1 2903471	61	0 9070654	94	0 8305504
29	1 2680795	62	0 9021398	95	0 8301607
30	1 2469152	63	0 8974405	96	0 8298465
31	1 2267826	64	0 8929597	97	0 8296059
32	1 2076173	65	0 8886889	98	0 8294366
33	1 1893606	66	0 8846212	99	0 8293367
				1.00	0 8293038

This table has been carefully prepared by the aid of the 50th edition of the Bremiker—Vega tables. If any mistake should be fouud, please address me at Richmond Kentucky.

[In the heading of the above table, $\alpha = (1 + a + a^2)$ and $\beta = (a + a^2)$.]